Problem 8C,2
Suppose $\{a_k\}_{k\in\Gamma}$ is a family in $\mathbb{R}$ and $a_k \geq 0$ for each $k \in \Gamma$. Prove that the unordered sum $\sum_{k\in\Gamma} a_k$ converges if and only if

$$\sup\{\sum_{j\in\Omega} a_j : \Omega \; is\ a\ finite\ subset\ of \Gamma\} < \infty.$$

Furthermore, prove that if the sum $\sum_{k\in\Gamma} a_k$ converges then it equals the supremum in the above.

Proof. Denote the supremum by A. If $A < \infty$, for any $\epsilon > 0$, by definition of supremum we can find a finite subset Ω of Γ, such that

$$A - \epsilon < \sum_{j\in\Omega} a_j \leq A.$$

Then for any finite subset $\Omega^{'}$ ocntaining Ω, we have

$$A - \epsilon < \sum_{j\in\Omega} a_j \leq \sum_{j\in\Omega'} a_j \leq A$$

By definition 8.53, this means that the unordered sum $\sum_{k\in\Gamma} a_k$ converges and it equals A. Conversely, assume that the unordered sum $\sum_{k\in\Gamma} a_k$ converges, by definition 8.53, we can find $B \in (0, \infty)$ and finite subset Ω such that for any finite subset $\Omega^{'}$ containing Ω

$$|B - \sum_{j\in\Omega'} a_j| < 1$$

We claim $A \leq B + 1$. In fact, for any finite subset Ω_1, $\Omega \bigcup \Omega_1$ is also a finite subset and contains Ω. Thus

$$\sum_{j\in\Omega_1} a_j \leq \sum_{j\in\Omega\bigcup\Omega_1} a_j \leq B + 1$$

This proves the claim and thus finishes the proof. □

Problem 8C,6
Suppose $\{a_k\}_{k\in\Gamma}$ is a family in $\mathbb{R}$. Prove that the unordered sum $\sum_{k\in\Gamma} a_k$ converges if and only if $\sum_{k\in\Gamma} |a_k| < \infty$.

Proof. If $\sum_{k\in\Gamma} |a_k| < \infty$, for any positive integer k the set $\{a_k : |a_k| \in (\frac{1}{k+1}, \frac{1}{k}]\}$ must be a finite set. Thus $\{a_k : a_k \neq 0\}$ is a countable set. Thus the summation $\sum_{k\in\Gamma} a_k$ is in fact a countable summation. So by standard result, we know it converges.

Conversely, assume that the unordered sum $\sum_{k\in\Gamma} a_k$ converges. By definition 8.53 and argues as in problem 8C,2,

$$-\infty < \inf\{\sum_{j\in\Omega} a_j : \Omega \; is\ a\ finite\ subset\ of \Gamma\} \leq \sup\{\sum_{j\in\Omega} a_j : \Omega \; is\ a\ finite\ subset\ of \Gamma\} < \infty.$$

But if $\sum_{k\in\Gamma} |a_k| = \infty$, then one can obvioulsy show that either $\inf\{\sum_{j\in\Omega} a_j : \Omega \; is\ a\ finite\ subset\ of \Gamma\} = -\infty$ or $\sup\{\sum_{j\in\Omega} a_j : \Omega \; is\ a\ finite\ subset\ of \Gamma\} = \infty$. This implies that in fact $\sum_{k\in\Gamma} |a_k| < \infty$. □

Problem 8C,12
Prove the converse of Parseval's identity: if $\{e_k\}_{k\in\Gamma}$ is an orthonormal family in Hilbert space V, and for any $f \in V$,

$$\|f\|^2 = \sum_{k\in\Gamma} | < f, e_k > |^2,$$

then $\{e_k\}_{k\in\Gamma}$ is an orthonormal basis.

Proof. Note that one can easily check that $\|f - \sum_{k\in\Gamma} < f, e_k > e_k\|^2 = \|f\|^2 - \sum_{k\in\Gamma} | < f, e_k > |^2$. Then by assumption, the right hand side is zero and thus $f = \sum_{k\in\Gamma} < f, e_k > e_k$. □

Problem 8C,24
The Dirichlet space $\mathcal{D}$ is defind to the set of analytic function $f : D \to \mathbb{C}$ such that

$$\int_D |f^{'}|^2 d\lambda_2 < \infty.$$

For $f, g \in \mathcal{D}$, define $< f, g >= f(0)g(\bar{0}) + \int_D f^{'} \bar{g^{'}} d\lambda_2$

- Prove that $\mathcal{D}$ is a Hilbert space.
- Show that if $w \in D$, then $f \to f(w)$ is bounded linear functional on $\mathcal{D}$.
- Find an orthonomal basis of $\mathcal{D}$.
- Suppose $f \in \mathcal{D}$ has the Taylor series $f(z) = \sum_{k=0}^{\infty} a_k z^k$, find a formula for $\|f\|$ in terms of $a_0, a_1, \ldots$.
- Suppose $w \in D$, find an explict formula for $\Gamma_w \in \mathcal{D}$ such that for all $f \in \mathcal{D}$,

$$f(w) =< f, \Gamma_w >$$

Proof.
- Note that for any analytic function $f : D \to \mathbb{C}$, from the mean value property (which holds in generally for harmonic functions), we can get the following estimate:

$$|f(z)|^2 \leq \frac{1}{\sqrt{\pi}(1 - |z|)} \int_D |f|^2 d\lambda_2$$

  From the above estimate, we know that L_2 convergence of analytic functions implies the uniform convergence in every compact subset. From this fact, the completeness of D follows easily.
- This also follows the estimate above and definition.
- Normalize $1, z, z^2, \ldots$ to e_0, e_1, e_2 such that $\|e_k\| = 1$ for any nonnegative integer k. Then use Taylor series, these form a basis. In fact

$$e_0 = 1, e_k = \frac{z^k}{\sqrt{k\pi}} \; k \geq 1$$

- Use the result in 3, one can easily calculate

$$\|f\|^2 = |a_0|^2 + \sum_{k=1}^{\infty} |a_k|^2 k\pi$$

- Obviously such Γ_w is unique by Riesz representation theorem. If $w = 0$, then $\Gamma_0 = 1$. Assume now $w \neq 0$. Assume that $\Gamma_w(z) = \sum_{k=0}^{\infty} a_k z^k$. Take the test function f to be $z^k, k > 0$, we find that

$$w^k =< z^k, \Gamma_w >=< z^k, a_k z^k >= \bar{a_k} \pi k$$

  Thus for $k > 0$, $a_k = \frac{\bar{w}^k}{\pi k}$; aimilarily, $a_0 = 1$.
  □

Problem 9A,3
Suppose ν is a complex measure on a measurable space (X, S). Prove that $|\nu(X)| = \nu(X)$ if and only if ν is a positive measure.

Proof. Obviouly if ν is a positive measure, then $|\nu(X)| = |\nu|(X)$ by definition. Conversely, assume $|\nu(X)| = \nu(X)$. We can decompose $\nu = \nu_1 + i\nu_2$ as in the textbook. If both ν_1, ν_2 are nonzero, then one can find a measurable set $A \subset X$ such that $\nu(A) < |\nu|(A)$ strictly. And since we always have $\nu(X - A) \leq |\nu|(X - A)$. Thus $|\nu(X)| > \nu(X)$. This is a contradiction and so one of the ν_1, ν_2 must be zero. By assumption, obviouly we have that $\nu_2 = 0, \nu = \nu_1$ is a positive measure. □

Problem 9A,4
Suppose ν is a complex measure on a measurable space (X, S). Prove that if $E \in S$, then

$$|\nu|(E) = \sup\{\sum_{k=1}^{\infty} |\nu(E_k)| : E_1, E_2, \ldots \textit{ is disjoint sequence in } S \textit{ such that } E = \bigcup_{k=1}^{\infty} E_k\}$$

Proof. Denote the right hab=nd side by $A(E)$. Then obviously by definition, $\nu|(E) \leq A(E)$. Conversely, for any disjoint $E = \bigcup_{k=1}^{\infty} E_k$,

$$\sum_{k=1}^{\infty} |\nu(E_k)| = \lim_{N\to\infty} \sum_{k=1}^{N} |\nu(E_k)| \leq \lim_{N\to\infty} |\nu|(E) = |\nu|(E)$$

Then take the supremum over all such decomposition, we find $A(E) \leq |\nu|(E)$. This completes the proof. □

Problem 9A,11
For (X, S) a measurable space and $b \in X$, define a finite positiev measurable δ_b on (X, S) by $\delta_b(E) = 1 \ if \ b \in E; \delta_b(E) = 0 \ if \ b \notin E$.

- Show that if $b, c \in X$, then $\|\delta_b + \delta_c\| = 2$.
- Give an example of measurable space (X, S) and $b, c \in X$ such that $\|\delta_b - \delta_c\| \neq 2$.

Proof.
- In definition 9.8, just take $E_1 = X$ other sets to be empty. This easily show that $\|\delta_b + \delta_c\| \geq 2$. And it is trivial that $\|\delta_b + \delta_c\| \leq 2$. So the result follows.
- Just take X to be any set containg more than two elements and S contains only the empty set and X itself. Take any $b, c \in X$. Then $\|\delta_b - \delta_c\| = 0$.
□